Boolean Algebraic Structure on [0,1]

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Abstract: Firstly, we show that the famous problem of

“ Can the unit interval [0,1] be equipped with a Boolean algebraic structure ? ”

is an ill-posed problem. In this sense, we notice that the conventional answer for this problem

“ [0,1] can not be equipped with a Boolean algebraic structure ”

has no value in itself. We argue that we need additional conditions to answer the above problem legitimately.

Secondly, we propose a new non-standard model of real numbers.

Thirdly, we prove that, based on this new model, there is an algorithm to define a Boolean algebraic structure on [0,1].

§ 0. Introduction

There is an interesting problem in the field of AI, especially in the areas of, say, many-valued logics, expert systems, knowledge representation, reasoning, fuzzy theory, automated theorem proving and any topics with the probability and/or the (un)certainty etc. The conventional statement of the problem has the form of

“ Can the unit interval [0,1] be equipped with a Boolean algebraic structure ? ” $\cdots$(1)
or
“ Can we define a Boolean algebraic structure on the unit interval [0,1] ? ”

Why is this problem so interesting ? Because, the problem is closely related to the important issue of *the coherence of real-valued classical logic.* Here, the coherence means that

“ two equivalent formulas have the same value for any real-valued assignment of all atoms w.r.t. suitable logical operations $\wedge, \vee, \neg, \rightarrow$ etc over [0,1]. ”

Now, the first purpose of this paper is to claim that the above famous problem is a kind of *ill-posed problem*. Here, the notion of ill-posedness is a familiar one in the field of AI. In this case, we argue that the ill-posedness is located both in the interpretation of the unit interval [0,1] and in the realization methodology of operations $\wedge, \vee, \neg, \rightarrow$ etc.

The second purpose of this paper is to propose a novel non-standard model of real numbers, whose idea is profoundly concerned with the binary expression of real numbers.

The final and the main purpose of this paper is to show that we can positively solve the above ill-posed problem under the additional condition that

“ [0,1] is interpreted by the above non-standard model. ”

Precisely speaking, the result becomes the following form.

Theorem
We can concretely define functions c, d, n *which realize* $\wedge$, $\vee$, $\neg$ *respectively in the sense of* [0,1]*-valued classical logic, whenever* [0,1] *is interpreted by the non-standard model.*

§ 1. Ill-posedness

In order to claim the illposedness of the above famous problem (1), let's start our argument from a fundamental standpoint.

Suppose we are given the unit interval [0,1]. Then, what kind of semantics or interpretation can we attach to this syntactical expression [0,1]? One natural interpretation of [0,1] is that

“ it is a pure set of real numbers between 0 and 1 ” .

This kind of interpretation is employed in set theory, whenever we are interested in the cardinality of [0,1]. Another natural interpretation of [0,1] is that

“ it is a totally ordered set of real numbers between 0 and 1, where the ordering relation is the usual real number ordering ” .

This kind of interpretation is usually employed whenever we are concerned about the ordering relation of real numbers. Other natural interpretations of [0,1] are

“ it is a subset of the field of real numbers ”

from a viewpoint of algebra and

“ it is a subset of real line ”

from a viwpoint of topology. Of course, there is yet another interpretation from a viewpoint of analysis.

Thus, we notice that there can be many kind of semantics of the same syntactical expression [0,1], the unit interval! Each semantics depends on the situation we employ. Nevertheless, it is one point that the analytic semantis is the most complex among the above in the sense that it contains all the other [0,1]-semantics. From this viewpoint, at a first glance, it apparently seems that the analytic interpretation is sufficient to manage [0,1] in all areas of AI. However, the matter is not so simple. In the above, we did not mention about the logical interpretation of [0,1], which should be the base of our argument, and the interpretation is quite different from the analytic one.

Now, what is the logical interpretation of [0,1] in this context ? It should be the interpretation which is inevitably related to the (, to some extent, correct) inference or the reasoning using the property of [0,1]. And so, as the result, it is interested in the (lattice) ordering relation defined on [0,1] as a logical system. Here, the most important point in this argument is the fact that

“ in many cases, real number ordering does not become the required lattice ordering imposed by the expected logic ” .

To see this, just take an example of a fuzzy system whose operations for $(\wedge,\vee)$ are not (min,max).

By recognizing the above arguments, we notice that the conventional naive problem (1) is short of additional conditions and so, it is ill-posed. That is, to answer (1), we must fix the supposed interpretation of [0,1] which, at least, includes both analytic and logical semantics. In other words, we ought to declare what kind of analytical model of [0,1] and what kind of lattice-ordering over [0,1] we choose. In the next section, we show three typical claims which embody the ill-posedness of the problem (1).

§ 2. Three Claims

Throughout this section, we employ standard model of real analysis as the fixed analytic semantics for the sake of simplicity. Our first claim towards the positive answer for (1) has the following form.

Proposition 2-1. Existence Theorem
[0,1] *can be equipped with a Boolean algebraic structure by suitably choosing operations* c, d, n *for* $\wedge,\vee,\neg$ *respectively.*

Proof: Let B be a Boolean algebra whose cardinality is 2 . We can extend a 1:1, onto map $\eta:\mathrm{B}\rightarrow[0,1]$ such that $\eta(1)=1$ and $\eta(0)=0$ to a lattice isomorphism $\lambda:\mathrm{B}\cong[0,1]$ by suitably defining an ordering relation $<$ over [0,1]. Then, ([0,1],<) becomes a Boolean algebra. □

The remarkable point is that the above is a typical exitence theorem. That is, we can't obtain any concrete operations $(c,\ d,n)$ for $(\wedge,\vee,\neg)$ from the above positive result. First of all, at this point, we are not certain whether there are realization algorithms of $c,\ d,n$ over [0,1] or not. Concerning this aspect, we remember that the conventional answer for (1) has the following negative form

「[0,1] can't be equipped with a Boolean algebraic structure」. ⋯(2)
(See, for example, [4].)

Since we have already obtain Proposition 2-1, we notice that the above statement (2) is wrong, or at least, it lacks additional conditions. For example, the following form is a right statement.

Proposition 2-2.
[0,1] *can't be equipped with a Boolean algebraic structure if we choose real number ordering* $\leqq$ *as the lattice ordering.*

Proof: The linearly ordered Boolean algebra is the Boolean algebra $\mathbf{2}$ of two elements $\{0,1\}$ only. □

These two extreme and obvious (or even trivial) results Proposition 2-1 and Proposition 2-2 show the ill-posedness of (1).

Now, by Proposition 2-2, we notice that in order to define Boolean algebraic structure on [0,1], we must choose a logical semantics such that $(\wedge,\vee)$ is not (min,max), because real number ordering fixes

$$(\wedge,\vee)=(\min,\max)$$

Extending this negative result one step towards the frontier, we obtain

Theorem 2-3.
[0,1] *can't be equipped with a Boolean algebraic structure, if we choose T-norm (T-conorm) for* $\wedge$ $(\vee)$.

Proof: (I) the case of T-norm
Suppose that there is a T-norm c for $\wedge$ and an operation n for $\neg$ over [0,1] such that $([0,1],\mathrm{c},\mathrm{n})$ form a Boolean algebra. Our goal is to reach a contradiction. For this aim, let r be a real number s.t. $0<r<1$ and consider $\mathrm{n}(r)$. Suppose $\mathrm{n}(r)=1$, then

$$r=\mathrm{n}(\mathrm{n}(r))=\mathrm{n}(1)=0.$$

This means that $\mathrm{n}(r)\neq 1$. Similarly, $\mathrm{n}(r)\neq 0$. So, we notice that
$0<\mathrm{n}(r)<1$.

Now, suppose $r\leqq \mathrm{n}(r)$, where $\leqq$ is the real number ordering. Here, consider $\mathrm{c}(r,r)$ and $\mathrm{c}(\mathrm{n}(r),r)$. Since $([0,1],\mathrm{c},\mathrm{n})$ is a Boolean algebra, $\mathrm{c}(r,r)=r$ and $\mathrm{c}(\mathrm{n}(r),r)=0$. Thus,

$\mathrm{c}(r,r)>\mathrm{c}(\mathrm{n}(r),r)$.

This contradicts the condition that c is a T-norm, because any T-norm t must satisfy the following monotonic condition that,

for any real numbers $0\leqq x\leqq u\leqq 1$ and $0\leqq y\leqq v\leqq 1$, $\mathrm{t}(x,y)\leqq \mathrm{t}(u,v)$. $\cdots$(3)

Similarly, from the other supposition of $r>\mathrm{n}(r)$, we get a contradiction via $\mathrm{n}(r)=\mathrm{c}(\mathrm{n}(r),\mathrm{n}(r))>\mathrm{c}(\mathrm{n}(r),r)=0$.

(II) the case of T-conorm is similar. □

From this result, we recognize that, to obtain Boolean algebra over [0,1], neither c for $\wedge$ must be T-norm nor d for $\vee$ must be T-conorm. However, to define a Boolean algebraic structure on [0,1], we need not restrict our attention to T-norms and T-conorms. To tell the truth, in order to define a suitable lattice ordering over [0,1] by using operations (c,d) for $(\wedge,\vee)$, they must satisfy the following idempotent condition as far as $(\wedge,\vee)$ is interpreted as (*sup*,*inf*) of the lattice ordering.

For any real number $0\leqq r\leqq 1$,
$\mathrm{c}(r,r)=\mathrm{d}(r,r)=r$.

This is because $\wedge$ and $\vee$ should be interpreted as inf and sup of the required lattice ordering from a logical semantics of [0,1].

Remark: *You may interprete* $\wedge$, *say, as* $\times$ *over* [0,1]. *As the result,* $\mathrm{c}(0.5,0.5)$ *becomes 0.25. Well, this is one thing. However, the fact that the formula* $A\wedge A$ *does not become equivalent to A by the interpretation is another. The point is that you must* *propose a reasonable meaning of the interpretation!* ⊣

From the above observations, it is worth defining the next notion.

Definition 2-4.
(i) A binary operation b over [0,1] is called " B-norm " if it satisfies
1. b(0,0)=0 and b(r,1)=b(1,r)=r for all $r\in[0,1]$ (Boundary Condition)
2. b(r,s)=b(s,r) for all $r,s\in[0,1]$ (Commutativity)
3. b(r,b(s,t))=b(b(r,s),t) for all $r,s,t\in[0,1]$ (Associativity)
4. b(r,r)=r for all $r\in[0,1]$ (Idempotentness)

(ii) A binary operation b over [0,1] is called " B-conorm " if it satisfies
1'. b(1,1)=1 and b(r,0)=b(0,r)=r for all $r\in[0,1]$ (Boundary Condition)
and the above conditions 2, 3, 4. ⊣

Thus, the notion of B-norm (B-conorm) is obtained from the notion of T-norm (T-conorm) by exchanging the monotonic condition (3) to the idempotentness 4. Using this definition, we notice that

Proposition 2-5.
In order to define Boolean algebraic structure on [0,1], *we must choose operations of both B-norm for* $\wedge$ *and B-conorm for* $\vee$.

Proof: Easy to check. □

Now, what kind of B-norms and B-conorms should we choose ? At this moment, the answer is open for the standard model of [0,1]. However, from a viewpoint of computer science, there is a quite interesting semantics of [0,1] based on the non-standard model of real numbers. In the following, we investigate the topic.

§ 3. Non-standard Binary Model of Real Numbers

In the following, we choose [0,1] instead of the whole set of real numbers R as the target domain. (To argue R, it is enough to transfer [0,1] by integers.) The aim of this section is to propose a new model of [0,1] based on the digital and the discrete world view.

Now, suppose the situation that we would like to represent a real number r∈[0,1] by the binary notation. We already know that this kind of expression is familiar in the field of information science. Here, there is one point which is worth noting. The point is the fact that any real number which is generated by a sum of finite subset of $\{(1/2)^n \mid n = 1,2,\cdots\}$ can be expressed by two different infinite binary decimals. For example, in the case of 0.5, we can expree it by either

0.1000··· ···(4)

or

0.0111··· ···(5)

To be more precise, from a viewpoint of standard model of real numbers, these two binary decimal expressions are suppoed to represent the same real number 0.5 based on the completeness property of real numbers. (The Convergence of Infinite Series)

On the other hand, from a viewpoint of digital expression based on {0,1}, these two are different objects. In other words, they are syntactically different expressions.

Now, standing on the philosophy of knowledge representation, we recognize the following fact. Depending on the situations we choose, or even in the same context, there often happen the cases that the object as a semantical substance (in this case, the same real number 0.5) and the object as the syntactical expression (in this case, different expressions (4) and (5)) are mixed and confusedly used. In these cases, there sometimes occurs the state that we want to or need to distinguish syntactically different expressions as different objects. Especially when we are encountered the case that essentially infinite expressions are approximated by finite abbreviations, the necessity increases. (In this case, infinite decimals are approximated by finite decimals inside the computer.)

Grounding on these general observations, we employ the following idea as the foundation of our model of [0,1].

「Since there happen the cases that we had better distinguish different syntactical expressions as different objects, it is reasonable to employ a model of [0,1] which can inherently interprete two different expressions (4) and (5) as different entities.」

Based on this idea, we can define a new model of [0,1] by the following way.

Ⅰ Take the set

$$D = \{0.p_1p_2\cdots p_n\cdots \mid p_n \in \{0,1\}\}$$

of binary infinite decimals as the universe of the model.

Ⅱ Define two equivalence relations $=$, $\equiv$ and two ordering relations $\prec$, $\leqq$ over D so that

(i) $=$ is the syntactical identity over D, i.e.,

$0.p_1p_2\cdots p_n\cdots = 0.q_1q_2\cdots q_n\cdots$

iff

$p_n = q_n$ for all n

(ii) $\equiv$ is the semantical identity based on the standard model of [0,1],

i.e.,

$= \subset \equiv$

and

$0.p_1p_2\cdots p_n1000\cdots \equiv 0.p_1p_2\cdots p_n0111\cdots$

for all finite decimal

$0.p_1p_2\cdots p_n$.

(iii) $\prec$ is the lexicographical ordering over D, i.e.,

$0.p_1p_2\cdots p_n\cdots \prec 0.q_1q_2\cdots q_n\cdots$

iff

$p_1 \leqq q_1$ or ($p_1 = q_1$ and $p_2 \leqq q_2$) or $\cdots$

where $\leqq$ is the natural number ordering on {0,1}.

(iv) $\leqq$ is the semantical (real number) ordering based on the standard model

of [0,1], i.e.,

$\prec \subset \leqq$

and

$0.p_1p_2\cdots p_n1000\cdots \leqq 0.p_1p_2\cdots p_n0111\cdots$

for all finite decimal

$0.p_1 p_2 \cdots p_n$.

Then, it is easy to check that $(\mathrm{D}, =, \equiv, \prec, \leqq)$ becomes a model of [0,1] by suitably defining other operations $+$, $\times$ etc based on the standard model of [0,1]. $((\mathrm{D}/\equiv, =', \leqq')$ becomes the standard model of [0,1], where $='$ is the identity over $\mathrm{D}/\equiv$ and $\leqq'$ is the real number ordering corresponding to $\leqq$.)

The essential difference between the above new model and the conventional standard model is that $='$ and $\leqq'$ are defined by means of the syntactical relations $=$ and $\prec$ over binary infinite decimals. Clearly, this new approach is different from the conventional approach. In this sense, our model can be called "a new non-standard model" of real numbers.

Remark: *As is well-known, there is another non-standard model of real numbers based on the technique of the ultrapower method, i.e., the model which admits the notion of "infinitesimals" based on the non-standard analysis. (See, for example,* [6].*) Compared with this ultrapower model, our new model is not so different from the standard model. To tell the truth, it is rather controversial that which model is more standard from a viewpoint of (not mathematics but) information science, ours or the conventional model.* ⊣

In the following, for the sake of convenience, let's call the above new non-standard model "binary model" of real numbers.

§ 4. Boolean Algebraic Structure on [0,1] Based on Binary Model

The merit of fuzzy theory is in its non-standardness as a logical system. That is, in order to represent the fuzzyness of a proposition, the theory admits a variety of interpretations of logical symbols. As the result, to obtain the richness of the representation, the theory even permits other sets than mere [0,1] as the truth-value domain (, i.e., the range of membership functions). Since the matter is so, sticking to the standard model of [0,1] as the base of the interpretation can be said to contradict the philosophy of fuzzy theory. Thus, the interpretation of [0,1] based on the binary model ought to become legitimate from a viewpoint of the flexibility of operations. Especially when we are trying to quantize membership functions by the finite approximation, the legitimacy increases.

Based on this philosophy, we can concretely construct a Boolean algebraic structure on [0,1]. Our main claim becomes the following.

Theorem 4-1.

Based on the binary interpretation, we can algorithmically define (c, d, n) *for* ($\wedge, \vee, \neg$) *over* [0,1] *so that the resulting structure* ([0,1], c, d, n) *becomes a Boolean algebra.*

Proof: Let N be the set of all natural numbers and let $\mathrm{B} = (P(\mathrm{N}), \cap, \cup, \mathrm{N}-)$ be the complete Boolean algebra generated from the power set $P(\mathrm{N})$ of N. Then, it is well-known that the above B becomes isomorphic to the following structure ($2^{\mathrm{N}}, \omega\text{-}\wedge, \omega\text{-}\vee, \omega\text{-}\neg$), where

$$2^{\mathrm{N}} = \{f \mid f{:}\mathrm{N} \to 2\}$$

and, for any $f,g \in 2^{\mathrm{N}}$ and for all $n \in \mathrm{N}$,

$$\omega\text{-}\wedge(f,g)(n) = \min(f(n),g(n))$$
$$\omega\text{-}\vee(f,g)(n) = \max(f(n),g(n))$$
$$\omega\text{-}\neg(f)(n) = 1 - f(n).$$

By the way, there is a 1:1, onto map $\psi: 2^{\mathrm{N}} \to \mathrm{D}$, where D is the universe of binary model of [0,1] in the previous section. ψ is realized by the following correspondence. For any $f \in 2^{\mathrm{N}}$,

$$\psi(f) = 0.f(0)f(1)f(2)\cdots f(\mathrm{n})\cdots.$$

Then, based on the Boolean operations $\omega\text{-}\wedge, \omega\text{-}\vee, \omega\text{-}\neg$, we can define operations $\mathrm{c}, \mathrm{d}, \mathrm{n}$ on [0,1] so that ($2^{\mathrm{N}}, \omega\text{-}\wedge, \omega\text{-}\vee, \omega\text{-}\neg$) and ([0,1], $\mathrm{c}, \mathrm{d}, \mathrm{n}$) becomes isomorphic by the following way. For any $\mathrm{r,s} \in \mathrm{D}$,

$$\mathrm{c}(\mathrm{r,s}) = \psi(\omega\text{-}\wedge(\psi^{-1}(\mathrm{r}), \psi^{-1}(\mathrm{s})))$$
$$\mathrm{d}(\mathrm{r,s}) = \psi(\omega\text{-}\vee(\psi^{-1}(\mathrm{r}), \psi^{-1}(\mathrm{s})))$$
$$\mathrm{n}(\mathrm{r}) = \psi(\omega\text{-}\neg(\psi^{-1}(\mathrm{r}))).$$

Thus, we obtain a concrete algorithm to define a Boolean algebraic structure on [0,1] by using the above defined c, d, n. □

Three interesting properties of the above defined Boolean algebra are

(1) it is complete as a Boolean algebra
(2) c is a B-norm and d is a B-conorm
(3) the lattice ordering $<$ as a Boolean algebra is compatible with the real number ordering $\leqq$ on [0,1] in the sense that, for any $r,s \in D$,

$$r < s \rightarrow r \leqq s.$$

The checks are easy.

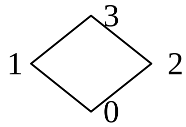
Practically speaking, the above realization algorithm gives the following methodology. Given arbitrary two real numbers $r,s \in [0,1]$. Translate r,s to binary infinite decimals (or sometimes, binary finite decimal approximations). The point is that we can choose one of two different binary decimals for the same real number which is generated by the finite subset of $\{(1/2)^n \mid n=1,2,\cdots\}$ at our disposal. This flexibility becomes the merit as fuzzy theory. Then, in order to define, say, $r \wedge s$, just use the above c or ω-$\wedge$, and the result is again translated to the usual real number.

§ 5. Conclusion

We have shown that the famous problem (1) is an ill-posed problem. Then we propose a condition, based on which (1) is positively solved. The condition is closely related to the binary expression of real numbers and it gives a new non-standard model of real numbers.

By the way, for any natural number $p \geqq 2$, we can define a non-standard model R_p of real numbers based on p-ary notation of real numbers by using the similar argument to the discussion in §3. Especially when $p=2^m$ for $m \in \mathbf{N}$, we can induce a Boolean algebraic structure on [0,1] by employing a suitable lattice ordering.

For example, when $m=2$, we can consider a Boolean algebra of the form



Based on this fact, by using the non-standard model R_4 of real numbers whose universe D_4 has the form $\{0.p_1p_2\cdots p_n\cdots \mid p_n\in\{0,1,2,3\}\}$, we obtain another concrete algorithm which realizes a Boolean algebraic structure on [0,1] via the similar argument to the proof of Theorem 4-1, by changing the definition of $\omega\text{-}\wedge, \omega\text{-}\vee, \omega\text{-}\neg$ to

$\omega\text{-}\wedge(f,g)(n) = f(n)\wedge_4 g(n)$ for all $n\in\mathbf{N}$
$\omega\text{-}\vee(f,g)(n) = f(n)\vee_4 g(n)$ for all $n\in\mathbf{N}$
$\omega\text{-}\neg(f)(n) = \neg_4 f(n)$ for all $n\in\mathbf{N}$

Here, $\wedge_4, \vee_4, \neg_4$ in the right hand side are the Boolean operations on 4.

Thus, we notice that there are infinitely many different realization algorithms of Boolean algebraic structure on [0,1], depending on each different non-standard model of real numbers. (More than one for each non-standard model R_{2^m} for $m\geqq 2$, by employing different lattice ordering on 2^m!) In this sense, the result of §4 is by no means trivial.

However, at this moment, whether there exists a concrete realization algorithm of Boolean algebraic structure on [0,1] based on the standard model of real numbers or not is open.

References

[1] Baldwin,J.F., Fuzzy Logic and Fuzzy Reasoning, *J.Man-Machine Studies* 11(1979), 465-480.

[2] Bell,R. and Giertz,M., On the Analytic Formalism of the Theory of Fuzzy Sets, *Information Sciences* 5(1973), 149-156.

[3] Dubois,D. and Prade,H., Fuzzy Sets in Approximate Reasoning, Part 1: Inference with Possibility Distributions, *Fuzzy Sets and Systems* 40(1991), 143-202.

[4] Dubois,D., Lang,J. and Prade,H., Fuzzy Sets in Approximate Reasoning, Part2:Logical Approaches, *Fuzzy Sets and Systems* 40(1991), 203-244.

[5] Giles,R., Lukasiewicz Logic and Fuzzy Set Theory, *J. Man-Machine Studies*

8(1976), 313-327.
[6] Robinson,A., *Non-standard Analysis*, North-Holland, 1966, 2nd edition 1974.
[7] Yamaguchi,J., Boolean [0,1]-valued Function, *Proc. of the Brazil-Japan Joint Symposium on Fuzzy Systems*, 61-67.
[8] Zadeh,L.A., Fuzzy Sets, *Information and Control* 8(1965), 338-353.